Math 522 Exam 1 Solutions

1. You're playing Fibonacci nim. You start with 66, and you go first. What are all your possible (initial) winning moves?

   We write $66 = 55 + 8 + 3$, as the sum of nonconsecutive Fibonacci numbers. We may take 3, leaving our opponent with a red position: $55+8$ and unable to take 8. Or, we may take 11, leaving our opponent with a red position: 55 and unable to take 55. All other moves end in disaster against a skilled opponent.

2. Recall that F_j stands for the j^{th} Fibonacci number. For all natural n, prove that $F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$.

   We proceed by induction. The base case is $n = 1$, which claims $F_1F_2 = F_2^2$. Since $F_1 = F_2 = 1$, this is true. Otherwise we assume the statement holds for n and try to prove it for $n + 1$. Note that the left hand side has $2n - 1$ terms, which increases by *two* when we increase n. Hence, we need to add *two* terms to each side, namely $F_{2n}F_{2n+1} + F_{2n+1}F_{2n+2}$. The right hand side simplifies as $F_{2n}^2 + F_{2n}F_{2n+1} + F_{2n+1}F_{2n+2} = F_{2n}(F_{2n} + F_{2n+1}) + F_{2n+1}F_{2n+2} = F_{2n}F_{2n+2} + F_{2n+1}F_{2n+2} = (F_{2n} + F_{2n+1})F_{2n+2} = F_{2n+2}^2$, as desired.